Problem 3.38

In an interesting version of the energy-time uncertainty principle[41] $\Delta t = \tau/\pi$, where τ is the time it takes $\Psi(x,t)$ to evolve into a state orthogonal to $\Psi(x,0)$. Test this out, using a wave function that is a linear combination of two (orthonormal) stationary states of some (arbitrary) potential: $\Psi(x,0) = \left(1/\sqrt{2}\right)[\psi_1(x) + \psi_2(x)]$.

Solution

The position-space wave function at $t = 0$ is a linear combination of two stationary states.

$$\Psi(x,0) = \frac{1}{\sqrt{2}}\psi_1(x) + \frac{1}{\sqrt{2}}\psi_2(x)$$

In order to obtain the position-space wave function at any later time t, multiply each term by the corresponding wiggle factor.

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\psi_1(x)e^{-iE_1t/\hbar} + \frac{1}{\sqrt{2}}\psi_2(x)e^{-iE_2t/\hbar}$$

Let τ be the time it takes for this wave function to become orthogonal to the initial state.

$$\begin{aligned}
0 &= \langle \Psi(x,\tau) \,|\, \Psi(x,0)\rangle \\
&= \int_{-\infty}^{\infty} \Psi^*(x,\tau)\Psi(x,0)\,dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2}}\psi_1^*(x)e^{iE_1\tau/\hbar} + \frac{1}{\sqrt{2}}\psi_2^*(x)e^{iE_2\tau/\hbar}\right]\left[\frac{1}{\sqrt{2}}\psi_1(x) + \frac{1}{\sqrt{2}}\psi_2(x)\right]dx \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{2}\psi_1^*(x)\psi_1(x)e^{iE_1\tau/\hbar} + \frac{1}{2}\psi_1^*(x)\psi_2(x)e^{iE_1\tau/\hbar} + \frac{1}{2}\psi_2^*(x)\psi_1(x)e^{iE_2\tau/\hbar} + \frac{1}{2}\psi_2^*(x)\psi_2(x)e^{iE_2\tau/\hbar}\right]dx \\
&= \frac{1}{2}e^{iE_1\tau/\hbar}\int_{-\infty}^{\infty}\psi_1^*(x)\psi_1(x)\,dx + \frac{1}{2}e^{iE_1\tau/\hbar}\int_{-\infty}^{\infty}\psi_1^*(x)\psi_2(x)\,dx \\
&\qquad + \frac{1}{2}e^{iE_2\tau/\hbar}\int_{-\infty}^{\infty}\psi_2^*(x)\psi_1(x)\,dx + \frac{1}{2}e^{iE_2\tau/\hbar}\int_{-\infty}^{\infty}\psi_2^*(x)\psi_2(x)\,dx \\
&= \frac{1}{2}e^{iE_1\tau/\hbar}(1) + \frac{1}{2}e^{iE_1\tau/\hbar}(0) + \frac{1}{2}e^{iE_2\tau/\hbar}(0) + \frac{1}{2}e^{iE_2\tau/\hbar}(1) \\
&= \frac{e^{iE_1\tau/\hbar} + e^{iE_2\tau/\hbar}}{2} \\
&= \frac{\left(\cos\frac{E_1\tau}{\hbar} + i\sin\frac{E_1\tau}{\hbar}\right) + \left(\cos\frac{E_2\tau}{\hbar} + i\sin\frac{E_2\tau}{\hbar}\right)}{2} \\
&= \frac{\left(\cos\frac{E_1\tau}{\hbar} + \cos\frac{E_2\tau}{\hbar}\right) + i\left(\sin\frac{E_1\tau}{\hbar} + \sin\frac{E_2\tau}{\hbar}\right)}{2}
\end{aligned}$$

[41] See L. Vaidman, *Am. J. Phys.* **60**, 182 (1992) for a proof.

Use the sum-to-product formulas for cosine-cosine and sine-sine.

$$
\begin{aligned}
0 &= \frac{2\cos\left(\frac{\frac{E_1\tau}{\hbar}+\frac{E_2\tau}{\hbar}}{2}\right)\cos\left(\frac{\frac{E_1\tau}{\hbar}-\frac{E_2\tau}{\hbar}}{2}\right)+2i\sin\left(\frac{\frac{E_1\tau}{\hbar}+\frac{E_2\tau}{\hbar}}{2}\right)\cos\left(\frac{\frac{E_1\tau}{\hbar}-\frac{E_2\tau}{\hbar}}{2}\right)}{2} \\
&= \cos\left[(E_1+E_2)\frac{\tau}{2\hbar}\right]\cos\left[(E_1-E_2)\frac{\tau}{2\hbar}\right]+i\sin\left[(E_1+E_2)\frac{\tau}{2\hbar}\right]\cos\left[(E_1-E_2)\frac{\tau}{2\hbar}\right] \\
&= \left\{\cos\left[(E_1+E_2)\frac{\tau}{2\hbar}\right]+i\sin\left[(E_1+E_2)\frac{\tau}{2\hbar}\right]\right\}\cos\left[(E_1-E_2)\frac{\tau}{2\hbar}\right] \\
&= \exp\left[i(E_1+E_2)\frac{\tau}{2\hbar}\right]\cos\left(|E_1-E_2|\frac{\tau}{2\hbar}\right)
\end{aligned}
$$

Since the exponential function is never zero,

$$
\begin{aligned}
\cos\left(|E_1-E_2|\frac{\tau}{2\hbar}\right) &= 0 \\
|E_1-E_2|\frac{\tau}{2\hbar} &= \frac{1}{2}(2n-1)\pi, \quad n = 0, \pm1, \pm2, \ldots \\
\tau &= \frac{(2n-1)\pi\hbar}{|E_1-E_2|}.
\end{aligned}
$$

The first time after $t = 0$ that $\Psi(x,t)$ becomes orthogonal to $\Psi(x,0)$ occurs when $n = 1$.

$$\tau_1 = \frac{\pi\hbar}{|E_1-E_2|}$$

As a result, using the prescribed definition for Δt,

$$\Delta t = \frac{\tau_1}{\pi} = \frac{\hbar}{|E_1-E_2|}.$$

The aim now is to determine $\Delta E = \sigma_H = \sqrt{\langle H^2\rangle - \langle H\rangle^2}$. By looking at the formula for $\Psi(x,t)$ in terms of the eigenstates, we see that the probabilities of measuring E_1 and E_2 are respectively

$$
\begin{aligned}
P(E_1) &= \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \\
P(E_2) &= \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\langle H\rangle &= \sum_{n=1}^{2} E_n P(E_n) = E_1P(E_1) + E_2P(E_2) = \frac{1}{2}E_1 + \frac{1}{2}E_2 \\
\langle H^2\rangle &= \sum_{n=1}^{2} E_n^2 P(E_n) = E_1^2P(E_1) + E_2^2P(E_2) = \frac{1}{2}E_1^2 + \frac{1}{2}E_2^2,
\end{aligned}
$$

which means the standard deviation in energy is

$$\begin{aligned}\Delta E &= \sqrt{\langle H^2\rangle - \langle H\rangle^2}\\ &= \sqrt{\left(\frac{1}{2}E_1^2 + \frac{1}{2}E_2^2\right) - \left(\frac{1}{2}E_1 + \frac{1}{2}E_2\right)^2}\\ &= \sqrt{\left(\frac{1}{2}E_1^2 + \frac{1}{2}E_2^2\right) - \left(\frac{1}{4}E_1^2 + \frac{1}{2}E_1E_2 + \frac{1}{4}E_2^2\right)}\\ &= \sqrt{\frac{1}{4}E_1^2 - \frac{1}{2}E_1E_2 + \frac{1}{4}E_2^2}\\ &= \sqrt{\frac{1}{4}\left(E_1 - E_2\right)^2}\\ &= \frac{|E_1 - E_2|}{2}.\end{aligned}$$

Confirm that the energy-time uncertainty principle is satisfied.

$$\Delta t\,\Delta E = \left(\frac{\hbar}{|E_1 - E_2|}\right)\left(\frac{|E_1 - E_2|}{2}\right) = \frac{\hbar}{2} \geq \frac{\hbar}{2}$$